Fourier Analysis on the Circle: Uniqueness, Convolution, and Summability

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Abstract

In this paper, we study several foundational ideas in Fourier analysis on the circle. We begin by asking a basic question of uniqueness: if all of the Fourier coefficients of an integrable function vanish, to what extent must the function itself vanish? We then introduce convolution as a natural way to rewrite Fourier partial sums and to organize many of the main arguments that follow. From there, we define good kernels and show how their concentration properties allow us to recover a function from suitable averages.

We next apply this framework to two important summability methods. First, we study Cesàro summability through the Fejér kernel, which gives a robust form of convergence even when ordinary convergence of Fourier series is delicate. We then turn to Abel summability and the Poisson kernel, which provide another method for recovering a function from its Fourier coefficients. Finally, we use the Poisson kernel to construct harmonic extensions of boundary data and connect these ideas to the Dirichlet problem in the unit disc.

We emphasize how uniqueness leads naturally to convolution, convolution leads to kernels, and kernels lead to summability and harmonic extension.

Introduction

Rather than studying a periodic function directly, we study the sequence of its Fourier coefficients and ask how much of the original function can be recovered from that frequency data. This leads to several natural questions. When do the Fourier coefficients determine the function uniquely? How should we interpret the partial sums of the Fourier series? And what can be done when ordinary convergence is too delicate to handle directly?

We develop a sequence of ideas that address those questions. We begin with a uniqueness result showing that if all Fourier coefficients of an integrable function vanish, then the function itself must vanish at each point of continuity. We then introduce convolution, which gives a convenient way to rewrite Fourier partial sums in terms of kernels. This point of view leads naturally to the notion of a good kernel, whose concentration properties make it possible to recover a function from suitable averages.

We then apply this framework to two important summability methods. The Fejér kernel gives Cesàro summability of Fourier series, while the Poisson kernel gives Abel summability. These methods show that even when ordinary convergence of Fourier series is subtle, one can still recover the original function in a robust sense. Finally, the Poisson kernel allows us to pass from Fourier series on the circle to harmonic functions in the unit disc, leading to the Dirichlet problem and its solution by Poisson integrals.

The first theorem begins with the most basic uniqueness question. The proof of the first theorem introduces a method that appears throughout the paper, namely testing a function against trigonometric polynomials or kernels that concentrate near a point.

1. Fourier Analysis on the Circle

Theorem 1.1. *Suppose that f is an integrable function on the circle with*

$$\widehat{f}(n) = 0 \qquad \textit{for all } n \in \mathbb{Z}.$$

Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.

Proof. We first deal with the case in which f is real-valued. We argue by contradiction. Assume there is a point θ_0 at which f is continuous and $f(\theta_0) \neq 0$. Replacing f by $-f$ if necessary, we may assume that

$$f(\theta_0) > 0.$$

By translating the variable, we may also assume without loss of generality that

$$\theta_0 = 0.$$

Thus f is continuous at 0 and $f(0) > 0$.

Our goal is to construct a sequence of trigonometric polynomials $\{p_k\}$ which become increasingly concentrated near 0, in such a way that

$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta)\, d\theta \to \infty \qquad \text{as } k \to \infty.$$

This will contradict the hypothesis that all Fourier coefficients of f vanish.

Since f is continuous at 0 and $f(0) > 0$, we can choose δ with

$$0 < \delta \leq \frac{\pi}{2}$$

such that

$$f(\theta) > \frac{f(0)}{2}$$

whenever $|\theta| < \delta$.

Now define

$$p(\theta) = \varepsilon + \cos\theta,$$

where $\varepsilon > 0$ is chosen small enough so that

$$|p(\theta)| < 1 - \frac{\varepsilon}{2}$$

whenever $\delta \leq |\theta| \leq \pi$. Next choose η with

$$0 < \eta < \delta$$

such that

$$p(\theta) \geq 1 + \frac{\varepsilon}{2}$$

whenever $|\theta| < \eta$. For each $k \geq 1$, let

$$p_k(\theta) = [p(\theta)]^k.$$

Since p is a trigonometric polynomial, each p_k is also a trigonometric polynomial.

Since $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, the integral of f against any trigonometric polynomial must vanish. Therefore,

$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta)\, d\theta = 0 \qquad \text{for every } k.$$

On the set $\{\theta : \delta \leq |\theta| \leq \pi\}$, we have

$$|p_k(\theta)| \leq \left(1 - \frac{\varepsilon}{2}\right)^k.$$

Therefore,

$$\left| \int_{\delta \leq |\theta| \leq \pi} f(\theta) p_k(\theta)\, d\theta \right| \leq \left(1 - \frac{\varepsilon}{2}\right)^k \int_{-\pi}^{\pi} |f(\theta)|\, d\theta.$$

On the other hand, for $|\theta| < \eta$ we have both

$$f(\theta) \geq \frac{f(0)}{2}$$

and

$$p_k(\theta) \geq \left(1 + \frac{\varepsilon}{2}\right)^k.$$

Hence

$$\int_{|\theta|<\eta} f(\theta)p_k(\theta)\,d\theta \geq 2\eta \cdot \frac{f(0)}{2}\left(1 + \frac{\varepsilon}{2}\right)^k.$$

Also, by our choice of δ, both $f(\theta)$ and $p_k(\theta)$ are nonnegative whenever $|\theta| < \delta$, so

$$\int_{\eta\leq|\theta|<\delta} f(\theta)p_k(\theta)\,d\theta \geq 0.$$

Therefore,

$$\int_{|\theta|<\delta} f(\theta)p_k(\theta)\,d\theta \geq 2\eta \cdot \frac{f(0)}{2}\left(1 + \frac{\varepsilon}{2}\right)^k.$$

Combining these estimates with the bound on the complementary region, we obtain

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)\,d\theta \geq 2\eta \cdot \frac{f(0)}{2}\left(1 + \frac{\varepsilon}{2}\right)^k - \left(1 - \frac{\varepsilon}{2}\right)^k \int_{-\pi}^{\pi} |f(\theta)|\,d\theta.$$

As $k \to \infty$, the right-hand side tends to ∞, which contradicts the fact that

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)\,d\theta = 0$$

for all k. This proves the theorem for real-valued f.

Now suppose f is complex-valued. Write

$$f(\theta) = u(\theta) + iv(\theta),$$

where u and v are real-valued. Then

$$u(\theta) = \frac{f(\theta) + \overline{f(\theta)}}{2}, \qquad v(\theta) = \frac{f(\theta) - \overline{f(\theta)}}{2i}.$$

The Fourier coefficients of both u and v vanish. Since f is continuous at θ_0, so are u and v. Applying the real-valued case to each of them gives

$$u(\theta_0) = v(\theta_0) = 0.$$

Hence $f(\theta_0) = 0$. $\square$

2. Convolutions

Given two 2π-periodic integrable functions f and g on $\mathbb{R}$, we define their convolution on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy.$$

Since the functions are periodic, a change of variables also gives the equivalent formula

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\,dy.$$

Loosely speaking, convolution behaves like a weighted average. For example, if $g \equiv 1$, then

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, dy,$$

so $f * g$ is constant and equal to the average value of f on the circle.

In the context of Fourier series, convolutions arise because the partial sums of the Fourier series of f can be written as

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{inx},$$

and hence

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left[\sum_{n=-N}^{N} e^{in(x-y)} \right] dy = (f * D_N)(x), \end{aligned}$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is the Nth Dirichlet kernel. Thus the problem of understanding the partial sums $S_N(f)$ reduces to understanding the convolution $f * D_N$.

Proposition 2.1. *Suppose that f, g, and h are 2π-periodic integrable functions. Then:*

(i) $f * (g + h) = (f * g) + (f * h)$.

(ii) $(cf) * g = c(f * g) = f * (cg)$ *for any* $c \in \mathbb{C}$.

(iii) $f * g = g * f$.

(iv) $(f * g) * h = f * (g * h)$.

(v) $f * g$ *is continuous.*

(vi) $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$ *for all* $n \in \mathbb{Z}$.

Proof. Properties (i) and (ii) follow immediately from the linearity of the integral.

We first prove the remaining statements under the additional assumption that f, g, and h are continuous. Since they are also 2π-periodic, all of them are bounded on $\mathbb{R}$.

To prove (vi), we compute

$$\widehat{f * g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g)(x) e^{-inx}\, dx.$$

Substituting the definition of convolution gives

$$\widehat{f * g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y)\, dy \right) e^{-inx}\, dx.$$

Since the integrand is continuous, we may interchange the order of integration. Making the change of variables $u = x - y$ and using periodicity gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x - y) e^{-in(x-y)}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(u) e^{-inu}\, du = \widehat{g}(n).$$

Therefore,

$$\widehat{f * g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \widehat{g}(n)\, dy = \widehat{f}(n)\widehat{g}(n).$$

To prove (iii), use a change of variables together with periodicity:

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y)\, dy = (g * f)(x).$$

For (iv),

$$((f * g) * h)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g)(t) h(x-t)\, dt.$$

Substituting the definition of $f * g$, interchanging the order of integration, and setting $u = t - y$ yields

$$((f * g) * h)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)(g * h)(x-y)\, dy = (f * (g * h))(x).$$

Finally, to prove (v), let $x_1, x_2 \in \mathbb{R}$. Then

$$(f * g)(x_1) - (f * g)(x_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \big[g(x_1 - y) - g(x_2 - y) \big]\, dy.$$

Since g is continuous and 2π-periodic, it is uniformly continuous. Thus, if $|x_1 - x_2| < \delta$,

$$|g(x_1 - y) - g(x_2 - y)| < \varepsilon$$

for every y. Therefore,

$$|(f * g)(x_1) - (f * g)(x_2)| \leq \frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(y)|\, dy.$$

This shows that $f * g$ is continuous.

This completes the proof in the continuous case. For general integrable functions, one uses approximation by continuous 2π-periodic functions so that the identities above pass to the limit. Hence (iii)–(vi) remain valid for arbitrary 2π-periodic integrable functions. □

3. Good Kernels

In the proof of the uniqueness theorem, we constructed a sequence of trigonometric polynomials $\{p_k\}$ that become increasingly concentrated near the origin, allowing us to detect the behavior of a function near a single point. This suggests a more general principle: if a family of kernels concentrates its mass near 0, then convolution with those kernels should recover the local behavior of a function.

Recall that the Nth partial sum of the Fourier series of f can be written as

$$S_N(f) = f * D_N,$$

where D_N is the Nth Dirichlet kernel. Thus questions about convergence of Fourier series can be reformulated as questions about convolution with families of kernels.

Definition 3.1. A family of kernels $\{K_n\}_{n=1}^{\infty}$ on the circle is called a *good kernel* if:

(a) For every $n \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1.$$

(b) There exists a constant $M > 0$ such that for every $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta \le |x| \le \pi} |K_n(x)|\, dx \to 0 \qquad \text{as } n \to \infty.$$

Remark 3.2. The Dirichlet kernels $\{D_N\}$ do not form a family of good kernels. Although

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)\, dx = 1,$$

they fail the uniform L^1 bound required in property (b). In fact,

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx$$

grows like a constant multiple of $\log N$. Thus convolution with D_N is not controlled in the same way as convolution with a good kernel.

This is one reason why pointwise convergence of the ordinary partial sums

$$S_N(f) = f * D_N$$

is more delicate than convergence of the averaged sums. It motivates replacing ordinary convergence by summability methods, such as Cesàro summability with the Fejér kernel and Abel summability with the Poisson kernel.

Lemma 3.3. *Let $f \in L^1([-\pi, \pi])$, and let x_0 be a point at which f is continuous. Then for every $\varepsilon > 0$, there exists a continuous 2π-periodic function g such that*

$$\int_{-\pi}^{\pi} |f(y) - g(y)|\, dy < \varepsilon$$

and

$$g(x_0) = f(x_0).$$

Remark 3.4. The lemma is a consequence of the density of continuous functions in $L^1([-\pi, \pi])$, together with a local adjustment near the point x_0.

Theorem 3.5. *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and let f be an integrable function on the circle. Then*

$$\lim_{n \to \infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere on the circle, then the convergence is uniform.

Proof. Fix a point x at which f is continuous, and let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that

$$|y| < \delta \implies |f(x - y) - f(x)| < \varepsilon.$$

Using property (a),

$$(f * K_n)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) \big[f(x - y) - f(x) \big]\, dy.$$

Therefore,

$$|(f * K_n)(x) - f(x)| \le \frac{1}{2\pi} \int_{|y| < \delta} |K_n(y)|\, |f(x - y) - f(x)|\, dy$$
$$+ \frac{1}{2\pi} \int_{\delta \le |y| \le \pi} |K_n(y)|\, |f(x - y) - f(x)|\, dy.$$

On $|y| < \delta$, property (b) gives the bound

$$\frac{M\varepsilon}{2\pi}.$$

For the complementary region, the boundedness used in the paper gives a constant $B > 0$ such that

$$|f(x-y) - f(x)| \leq 2B,$$

and hence

$$\frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy \leq \frac{2B}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)|\,dy.$$

By property (c), this term tends to zero. Thus

$$\limsup_{n\to\infty} |(f * K_n)(x) - f(x)| \leq \frac{M\varepsilon}{2\pi}.$$

Since $\varepsilon > 0$ was arbitrary, $(f * K_n)(x) \to f(x)$.

If f is continuous everywhere on the circle, then f is uniformly continuous, so δ may be chosen independently of x. Repeating the same estimate and taking the supremum over x gives uniform convergence. ☐

4. Cesàro and Abel Summability: Applications to Fourier Series

Since Fourier series may fail to converge at individual points, we can try to recover the function by interpreting ordinary convergence in a weaker sense.

4.1. Cesàro Means and Summation

Suppose we are given a series of complex numbers

$$\sum_{k=0}^{\infty} c_k.$$

Its nth partial sum is

$$s_n = \sum_{k=0}^{n} c_k.$$

Ordinary convergence means that $s_n \to s$ as $n \to \infty$.

A basic example is the alternating series

$$1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k,$$

whose partial sums are

$$1, 0, 1, 0, \ldots$$

and therefore the series does not converge. Nevertheless, these partial sums oscillate evenly between 1 and 0, so it is natural to regard their average as the more meaningful limiting value.

Definition 4.1. Given a sequence of partial sums $\{s_k\}$, the Nth Cesàro mean is defined by

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

If $\sigma_N \to \sigma$ as $N \to \infty$, we say that the series is Cesàro summable to σ.

For the alternating series above, one checks directly that

$$\sigma_N \to \frac{1}{2}.$$

Theorem 4.2. *If the series*

$$\sum_{k=0}^{\infty} c_k$$

converges in the usual sense to s, then it is also Cesàro summable to s.

Proof. Let

$$s_n = \sum_{k=0}^{n} c_k$$

and suppose $s_n \to s$. By definition,

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

Fix $\varepsilon > 0$. Since $s_n \to s$, there exists N_0 such that

$$|s_n - s| < \varepsilon \qquad \text{whenever } n \geq N_0.$$

Then

$$|\sigma_N - s| \leq \frac{1}{N} \sum_{k=0}^{N_0-1} |s_k - s| + \frac{1}{N} \sum_{k=N_0}^{N-1} |s_k - s|.$$

For the second sum,

$$\frac{1}{N} \sum_{k=N_0}^{N-1} |s_k - s| \leq \frac{N - N_0}{N} \varepsilon \leq \varepsilon.$$

Let

$$C = \sum_{k=0}^{N_0-1} |s_k - s|.$$

Then the first term equals $C/N \to 0$. Hence, for sufficiently large N,

$$|\sigma_N - s| \leq \frac{C}{N} + \varepsilon < 2\varepsilon.$$

Therefore $\sigma_N \to s$. $\square$

4.2. Fejér's Theorem

Recall that

$$S_n(f) = f * D_n.$$

Define the Nth Cesàro mean of the Fourier series of f by

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + \cdots + S_{N-1}(f)(x)}{N}.$$

Since convolution is linear,

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where

$$F_N(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{N-1}(x)}{N}$$

is called the Nth Fejér kernel.

Lemma 4.3. *For every $N \geq 1$,*

$$F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)} \qquad (x \neq 0),$$

and this expression extends continuously to $x = 0$ by setting

$$F_N(0) = N.$$

Moreover, the family $\{F_N\}$ is a family of good kernels.

Proof. For $x \neq 0$, the identity is the standard trigonometric computation obtained from the formula for the Dirichlet kernels. At $x = 0$,

$$F_N(0) = \frac{D_0(0) + \cdots + D_{N-1}(0)}{N} = \frac{1 + 3 + \cdots + (2N-1)}{N} = N.$$

Since each D_n satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x)\, dx = 1,$$

we also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x)\, dx = 1.$$

The explicit formula shows that $F_N(x) \geq 0$, so

$$\int_{-\pi}^{\pi} |F_N(x)|\, dx = 2\pi.$$

Finally, fix $\delta > 0$. On

$$E_\delta = \{x \in [-\pi, \pi] : \delta \leq |x| \leq \pi\},$$

$\sin^2(x/2)$ has a positive minimum c_δ. Hence

$$F_N(x) \leq \frac{1}{N c_\delta}$$

on E_δ, and therefore

$$\int_{\delta \leq |x| \leq \pi} |F_N(x)|\, dx \leq \frac{2\pi}{N c_\delta} \to 0.$$

Thus $\{F_N\}$ is a family of good kernels. □

Theorem 4.4 (Fejér's Theorem)**.** *If f is integrable on the circle, then the Fourier series of f is Cesàro summable to f at every point of continuity of f. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesàro summable to f.*

Proof. Since

$$\sigma_N(f) = f * F_N$$

and the Fejér kernels form a family of good kernels, the conclusion follows from the good-kernel convergence theorem. □

Corollary 4.5. If f is integrable on the circle and

$$\hat{f}(n) = 0 \qquad \text{for all } n \in \mathbb{Z},$$

then $f = 0$ at every point of continuity of f.

Proof. If all Fourier coefficients vanish, then every partial sum $S_N(f)$ is identically zero. Hence every Cesàro mean $\sigma_N(f)$ is also identically zero. By Fejér's theorem, $\sigma_N(f)(x) \to f(x)$ at every point of continuity, so $f(x) = 0$ at each such point. □

Corollary 4.6. Every continuous function on the circle can be uniformly approximated by trigonometric polynomials.

Proof. If f is continuous, then by Fejér's theorem,

$$\sigma_N(f) \to f$$

uniformly on the circle. But each $\sigma_N(f)$ is a finite average of partial sums of the Fourier series, and each partial sum is a trigonometric polynomial. Therefore each $\sigma_N(f)$ is itself a trigonometric polynomial. □

4.3. Abel Means and Summation

There is another method of summation, due to Abel, which is even more powerful than Cesàro summation.

Definition 4.7. A series

$$\sum_{k=0}^{\infty} c_k$$

is said to be Abel summable to s if for every $0 \leq r < 1$, the power series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1^-} A(r) = s.$$

The quantities $A(r)$ are called the Abel means of the series.

As in the Cesàro case, ordinary convergence implies Abel summability to the same limit. In fact Abel summability is strictly stronger than Cesàro summability.

For example,

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1)$$

is Abel summable to 1/4, since

$$\sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2},$$

and therefore

$$\lim_{r \to 1^-} \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{4}.$$

4.4. The Poisson Kernel and Dirichlet's Problem in the Unit Disc

To adapt Abel summability to Fourier series, define the Abel means of the Fourier series

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}, \qquad 0 \leq r < 1.$$

Since f is integrable, the Fourier coefficients a_n are bounded, so this series converges absolutely and uniformly for each fixed $r < 1$.

Define the Poisson kernel by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Then

$$A_r(f)(\theta) = (f * P_r)(\theta).$$

Lemma 4.8. *If $0 \leq r < 1$, then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Moreover, as $r \to 1^-$, the Poisson kernels form a family of good kernels.

Proof. The displayed identity is the standard closed form for the Poisson kernel.

First, $P_r(\theta) \geq 0$. Integrating the defining Fourier series term by term gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1.$$

Since $P_r \geq 0$,

$$\int_{-\pi}^{\pi} |P_r(\theta)|\, d\theta = 2\pi.$$

Finally, fix $\delta > 0$. We use

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta).$$

On

$$E_\delta = \{\theta \in [-\pi, \pi] : \delta \leq |\theta| \leq \pi\},$$

there exists $c_\delta > 0$ such that

$$1 - \cos\theta \geq c_\delta.$$

Hence, if $1/2 \leq r < 1$,

$$1 - 2r\cos\theta + r^2 \geq c_\delta$$

on E_δ. Therefore

$$P_r(\theta) \leq \frac{1-r^2}{c_\delta},$$

and

$$\int_{\delta \leq |\theta| \leq \pi} |P_r(\theta)|\, d\theta \leq \frac{2\pi(1-r^2)}{c_\delta} \to 0 \qquad \text{as } r \to 1^-.$$

Thus the Poisson kernels form a family of good kernels. □

Theorem 4.9. *The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity of f. Moreover, if f is continuous on the circle, then the Fourier series is uniformly Abel summable to f.*

Proof. Since

$$A_r(f) = f * P_r$$

and the Poisson kernels form a family of good kernels as $r \to 1^-$, the result follows directly from the good-kernel convergence theorem. □

We now return to the Dirichlet problem in the unit disc. Motivated by the Fourier expansion, define

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta},$$

where a_m is the mth Fourier coefficient of f. Equivalently,

$$u(r, \theta) = A_r(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) P_r(\theta - \phi)\, d\phi.$$

Theorem 4.10. *Let f be an integrable function on the unit circle, and define*

$$u(r, \theta) = (f * P_r)(\theta).$$

Then:

(i) u has two continuous derivatives in the unit disc and satisfies

$$\Delta u = 0.$$

(ii) If θ is a point of continuity of f, then

$$\lim_{r \to 1^-} u(r, \theta) = f(\theta).$$

If f is continuous everywhere, then the convergence is uniform in θ.

(iii) If f is continuous, then u is the unique solution of the Dirichlet problem in the disc with boundary values f.

Proof. For part (i), recall that

$$u(r, \theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}.$$

Fix $\rho < 1$. On the closed disc $0 \leq r \leq \rho$, the series for u and all of its termwise derivatives converge absolutely and uniformly, because the factors $r^{|m|}$ give geometric decay. Therefore u may be differentiated term by term inside the unit disc.

In polar coordinates,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Applying this operator term by term to

$$r^{|m|} e^{im\theta}$$

shows that each term is harmonic, hence

$$\Delta u = 0.$$

Part (ii) follows from Abel summability via the Poisson kernel. For continuous boundary data, part (iii) gives the corresponding solution of the Dirichlet problem. $\square$

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