A Survey on Transport Proofs of Discrete Variants of the Prékopa–Leindler Inequality

Lloyd Sangwoo Ko
Brown University
sangwoo_ko@brown.edu

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Abstract

This report studies the paper *Transport Proofs of Some Discrete Variants of the Prékopa–Leindler Inequality* by Gozlan, Roberto, Samson, and Tetali. The report explains how ideas from entropy and transport can be used to prove two discrete functional inequalities, the Four Functions Theorem on the hypercube and the Klartag–Lehec inequality on the integers. Both inequalities arise from a stronger entropic statement, namely a discrete displacement convexity of entropy on $\mathbb{Z}$. The paper first introduces the necessary preliminaries. Then the report explains the transport proof of the Four Functions Theorem and shows how the entropy inequality on $\mathbb{Z}$ leads to the Klartag–Lehec inequality. Finally, the report briefly surveys the paper's extension to curved Brunn–Minkowski inequalities.

1. Introduction

The classical Prékopa–Leindler inequality is a fundamental result in convex geometry, probability, and analysis. It is closely related to convexity, log-concavity, and the geometry of measures. The paper of Gozlan, Roberto, Samson, and Tetali (2019) asks what remains true in discrete settings, where the usual transport tools from the continuous case are no longer directly available.

More specifically, the Prékopa–Leindler inequality is often viewed as a functional form of the Brunn–Minkowski inequality from convex geometry. The Brunn–Minkowski inequality describes how the volume of Minkowski sums of sets behaves and is one of the foundational results of convex geometry. In particular, it implies powerful structural results about convex bodies and log-concave measures. The Prékopa–Leindler inequality extends this geometric statement to functions and plays a central role in probability, concentration of measure, and functional inequalities. Many important results in analysis, including log-Sobolev inequalities and concentration inequalities, are closely connected to this framework. Before the discrete setting, transport proofs of Prékopa–Leindler had already been extended to continuous curved spaces. McCann (1997) introduced displacement convexity of entropy in Euclidean settings, while Cordero-Erausquin, McCann, and Schmuckenschläger (2001) extended Prékopa–Leindler to Riemannian manifolds with lower Ricci curvature bounds. Later work of Otto–Villani (2000), von Renesse–Sturm (2005), and the Lott–Sturm–Villani theory (2006) connected entropy convexity to lower Ricci curvature bounds on metric measure spaces. The discrete results of Gozlan, Roberto, Samson, and Tetali (2019) can be viewed as a discrete analogue of this program on spaces such as the hypercube and $\mathbb{Z}$.

Because of this, understanding whether similar inequalities remain true in discrete spaces is a natural and important question. However, discrete spaces such as the hypercube or the integers do not have the same geometric structure as Euclidean space, and midpoint operations are no longer straightforward. The paper addresses exactly this problem by developing discrete transport methods that recover analogues of these classical inequalities.

The paper studies two discrete analogues. The first is the Four Functions Theorem on the hypercube

$$\Omega_n = \{0,1\}^n,$$

where

$$x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n)), \qquad x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n)).$$

It states that if $f, g, h, k : \Omega_n \to \mathbb{R}_+$ satisfy

$$f(x)g(y) \leq h(x \wedge y)k(x \vee y) \qquad \text{for all } x, y \in \Omega_n,$$

then

$$\left(\sum_{x \in \Omega_n} f(x)\right) \left(\sum_{x \in \Omega_n} g(x)\right) \leq \left(\sum_{x \in \Omega_n} h(x)\right) \left(\sum_{x \in \Omega_n} k(x)\right).$$

The second is the Klartag–Lehec inequality on $\mathbb{Z}$. Writing

$$m^-(x, y) = \left\lfloor \frac{x+y}{2} \right\rfloor, \qquad m^+(x, y) = \left\lceil \frac{x+y}{2} \right\rceil,$$

it states that if $f, g, h, k : \mathbb{Z} \to \mathbb{R}_+$ satisfy

$$f(x)g(y) \leq h(m^-(x, y))k(m^+(x, y)) \qquad \text{for all } x, y \in \mathbb{Z},$$

then

$$\left(\sum_{x \in \mathbb{Z}} f(x)\right) \left(\sum_{y \in \mathbb{Z}} g(y)\right) \leq \left(\sum_{x \in \mathbb{Z}} h(x)\right) \left(\sum_{y \in \mathbb{Z}} k(y)\right).$$

The main contribution of the paper is that these inequalities are explained through a common entropic transport principle. The paper develops the necessary preliminaries, then proves the key one-dimensional transport argument on the hypercube, and explains how the entropic theorem on $\mathbb{Z}$ implies the Klartag–Lehec inequality.

2. Preliminaries

2.1. Pushforwards and couplings

If μ is a probability measure on a set E and $T : E \to F$ is a map, the pushforward measure $T_\#\mu$ on F is defined by

$$T_\#\mu(B) = \mu(T^{-1}(B)) \qquad \text{for all } B \subseteq F.$$

Equivalently, if X has law μ, then $T(X)$ has law $T_\#\mu$. To understand this intuitively with a discrete example, consider rolling a fair six-sided die, so $E = \{1, 2, 3, 4, 5, 6\}$ and $\mu(x) = 1/6$ for all x. If our map $T(x) = x \bmod 2$ maps the outcomes to $F = \{0, 1\}$, the pushforward measure $T_\#\mu$ describes the parity of the roll, yielding $T_\#\mu(0) = 1/2$ and $T_\#\mu(1) = 1/2$. If ν_1 and ν_2 are probability measures on E_1 and E_2, a coupling of ν_1 and ν_2 is a probability measure π on $E_1 \times E_2$ whose marginals are ν_1 and ν_2. In the discrete setting, couplings are often more natural than transport maps, since mass may need to be split. For instance, if ν_1 and ν_2 are both fair coin flips on $\{0, 1\}$, a valid coupling is the independent coupling $\pi(x, y) = \nu_1(x)\nu_2(y) = 1/4$ for all pairs, while another valid coupling is the deterministic identity coupling where $\pi(0, 0) = 1/2$ and $\pi(1, 1) = 1/2$.

2.2. Relative entropy

Let μ and ν be probability measures on a finite or countable set E, with ν absolutely continuous with respect to μ. The relative entropy of ν with respect to μ is

$$H(\nu \mid \mu) = \sum_{x \in E} \nu(x) \log\left(\frac{\nu(x)}{\mu(x)}\right).$$

If $d\nu = Z\, d\mu$, then

$$H(\nu \mid \mu) = \int Z \log Z\, d\mu.$$

So relative entropy is the entropy of the density of ν relative to the reference measure μ. A particularly important case for this paper is the counting measure m on $\mathbb{Z}$. This measure assigns mass one to each integer:

$$m(\{x\}) = 1 \qquad \text{for every } x \in \mathbb{Z}.$$

More generally, for any finite set $A \subseteq \mathbb{Z}$,

$$m(A) = |A|.$$

Thus, when ν is a probability measure on $\mathbb{Z}$, the relative entropy becomes

$$H(\nu \mid m) = \sum_{x\in\mathbb{Z}} \nu(x) \log \nu(x).$$

2.3. A dual formula

Proposition 2.1. *Let E be a finite set, let μ be a probability measure on E, and let $\phi : E \to \mathbb{R}$. Then*

$$\log \int_E e^{\phi}\, d\mu = \sup_{\nu\in\mathcal{P}(E)} \left\{ \int_E \phi\, d\nu - H(\nu \mid \mu) \right\}.$$

Proof. Let ν be absolutely continuous with respect to μ, with density $f = d\nu/d\mu$. Set

$$Z = \int_E e^{\phi}\, d\mu, \qquad g = \frac{e^{\phi}}{Z}.$$

Then

$$\int_E \phi\, d\nu - H(\nu \mid \mu) = \int_E \phi f\, d\mu - \int_E f \log f\, d\mu = \log Z - \int_E f \log\left(\frac{f}{g}\right) d\mu.$$

The last term is a relative entropy, so it is nonnegative. Hence

$$\int_E \phi\, d\nu - H(\nu \mid \mu) \le \log Z.$$

Taking the supremum over ν gives one inequality, and equality holds when

$$\frac{d\nu^*}{d\mu} = \frac{e^{\phi}}{\int_E e^{\phi}\, d\mu}.$$

□

2.4. Monotone coupling and discrete midpoints

If ν_0 and ν_1 are probability measures on $\mathbb{R}$, their monotone coupling is obtained by matching quantiles. For $i = 0, 1$, let

$$F_{\nu_i}(t) = \nu_i((-\infty, t])$$

denote the cumulative distribution function (CDF) of ν_i. Let U be uniformly distributed on $(0, 1)$ and define

$$X_0 = F_{\nu_0}^{-1}(U), \qquad X_1 = F_{\nu_1}^{-1}(U).$$

The law of (X_0, X_1) is the monotone coupling of ν_0 and ν_1. Its support is ordered, so smaller points of one measure are paired with smaller points of the other. On $\mathbb{Z}$, the midpoint of two integers may fail to be an integer, so the paper instead uses

$$m^-(x, y) = \left\lfloor \frac{x+y}{2} \right\rfloor, \qquad m^+(x, y) = \left\lceil \frac{x+y}{2} \right\rceil.$$

If π is a coupling of ν_0 and ν_1, define

$$\nu^- = (m^-)_{\#}\pi, \qquad \nu^+ = (m^+)_{\#}\pi$$

as the two midpoint measures that appear in the entropic theorem.

3. The Four Functions Theorem

Theorem 3.1 (Four Functions Theorem (Ahlswede–Daykin, 1978))**.** *Let $h_1, h_2, h_3, h_4 : \Omega_n \to \mathbb{R}$ satisfy*

$$h_1(x) + h_2(y) \le h_3(x \wedge y) + h_4(x \vee y) \qquad \textit{for all } x, y \in \Omega_n.$$

Then

$$\left(\sum_{x\in\Omega_n} e^{h_1(x)}\right)\left(\sum_{x\in\Omega_n} e^{h_2(x)}\right) \le \left(\sum_{x\in\Omega_n} e^{h_3(x)}\right)\left(\sum_{x\in\Omega_n} e^{h_4(x)}\right).$$

3.1. A coupling lemma on Ω_1

Let $\Omega_1 = \{0, 1\}$ and define

$$S(x, y) = (x \wedge y, x \vee y).$$

Lemma 3.2. *Let ν_1, ν_2 be probability measures on Ω_1. If $\nu_2(0) \le \nu_1(0)$, then there exists a coupling π of ν_1 and ν_2 such that $S_\#\pi$ is also a coupling of ν_1 and ν_2. If $\nu_2(0) \ge \nu_1(0)$, then there exists a coupling π of ν_1 and ν_2 such that $S_\#\pi$ is a coupling of ν_2 and ν_1.*

Proof. The paper first proves the first case. Write $\pi(i, j)$ for the mass at $(i, j) \in \{0, 1\}^2$. Since π is a coupling,

$$\pi(0,0) + \pi(0,1) = \nu_1(0), \qquad \pi(0,0) + \pi(1,0) = \nu_2(0).$$

Furthermore,

$$S(0,0) = (0,0), \quad S(0,1) = (0,1), \quad S(1,0) = (0,1), \quad S(1,1) = (1,1).$$

So no point maps to $(1, 0)$. Therefore $S_\#\pi(1, 0) = 0$, which forces $\pi(1, 0) = 0$. Then

$$\pi(0,0) = \nu_2(0), \qquad \pi(0,1) = \nu_1(0) - \nu_2(0), \qquad \pi(1,1) = \nu_1(1).$$

This is correct because $\nu_2(0) \le \nu_1(0)$. The second case is similar. □

3.2. The one-dimensional proof

Proposition 3.3. *Let $h_1, h_2, h_3, h_4 : \Omega_1 \to \mathbb{R}$ satisfy*

$$h_1(x) + h_2(y) \le h_3(x \wedge y) + h_4(x \vee y) \qquad \textit{for all } x, y \in \Omega_1.$$

Then

$$\log \sum_{x\in\Omega_1} e^{h_1(x)} + \log \sum_{x\in\Omega_1} e^{h_2(x)} \le \log \sum_{x\in\Omega_1} e^{h_3(x)} + \log \sum_{x\in\Omega_1} e^{h_4(x)}.$$

Proof. Let m_1 be the uniform probability measure on Ω_1. By the dual formula,

$$\log \sum_{x\in\Omega_1} e^{h(x)} = \log 2 + \sup_{\nu\in\mathcal{P}(\Omega_1)} \left\{ \int h \, d\nu - H(\nu \mid m_1) \right\}.$$

Since the constant $\log 2$ cancels, it is enough to compare the variational terms. Let $\nu_1, \nu_2 \in \mathcal{P}(\Omega_1)$. Suppose $\nu_2(0) \le \nu_1(0)$. By the coupling lemma, there exists a coupling π of ν_1 and ν_2 such that $\pi' = S_\#\pi$ is also a coupling of ν_1 and ν_2. Then

$$\int h_1 \, d\nu_1 + \int h_2 \, d\nu_2 = \int_{\Omega_1^2} \big(h_1(x) + h_2(y)\big) \, d\pi(x, y).$$

Using the pointwise assumption,

$$\int_{\Omega_1^2} \big(h_1(x) + h_2(y)\big) \, d\pi(x, y) \le \int_{\Omega_1^2} \big(h_3(x \wedge y) + h_4(x \vee y)\big) \, d\pi(x, y).$$

By the definition of pushforward, this equals

$$\int h_3 \, d\nu_1 + \int h_4 \, d\nu_2.$$

Hence

$$\left(\int h_1 \, d\nu_1 - H(\nu_1 \mid m_1) \right) + \left(\int h_2 \, d\nu_2 - H(\nu_2 \mid m_1) \right)$$
$$\leq \left(\int h_3 \, d\nu_1 - H(\nu_1 \mid m_1) \right) + \left(\int h_4 \, d\nu_2 - H(\nu_2 \mid m_1) \right).$$

Optimizing over ν_1 and ν_2 yields the result. The case $\nu_2(0) \geq \nu_1(0)$ is analogous. $\square$

3.3. Induction to higher dimensions

To pass from Ω_1 to Ω_n, the paper uses induction. For $a \in \Omega_1$ and $h : \Omega_n \to \mathbb{R}$, define

$$h^a(x) = h(x, a), \qquad x \in \Omega_{n-1}.$$

Applying the induction hypothesis to the sliced functions produces functions

$$H_i(a) = \log \sum_{x \in \Omega_{n-1}} e^{h_i^a(x)}$$

on Ω_1, and these satisfy

$$H_1(a) + H_2(b) \leq H_3(a \wedge b) + H_4(a \vee b).$$

The one-dimensional case then yields the theorem on Ω_n.

4. Displacement convexity on $\mathbb{Z}$

The deeper theorem of the paper is an entropy inequality on $\mathbb{Z}$. Let ν_0 and ν_1 be probability measures on $\mathbb{Z}$ with compact support, let π be their monotone coupling, and define

$$\nu^- = (m^-)_\# \pi, \qquad \nu^+ = (m^+)_\# \pi.$$

Theorem 4.1 (Discrete Displacement Convexity of Entropy (Gozlan–Roberto–Samson–Tetali, 2019))**.** *If m denotes counting measure on $\mathbb{Z}$, then*

$$H(\nu^- \mid m) + H(\nu^+ \mid m) \leq H(\nu_0 \mid m) + H(\nu_1 \mid m).$$

That is, entropy is midpoint convex along the discrete interpolation determined by the monotone coupling.

4.1. Implication of the Klartag–Lehec inequality

Suppose

$$f(x)g(y) \leq h(m^-(x,y))k(m^+(x,y)) \qquad \text{for all } x, y \in \mathbb{Z}.$$

Taking logarithms and integrating with respect to the monotone coupling π of two probability measures ν_0 and ν_1 gives

$$\int \log f \, d\nu_0 + \int \log g \, d\nu_1 \leq \int \log h \, d\nu^- + \int \log k \, d\nu^+.$$

Using the entropy inequality,

$$H(\nu^- \mid m) + H(\nu^+ \mid m) \leq H(\nu_0 \mid m) + H(\nu_1 \mid m),$$

the paper obtains

$$\left(\int \log f \, d\nu_0 - H(\nu_0 \mid m)\right) + \left(\int \log g \, d\nu_1 - H(\nu_1 \mid m)\right)$$
$$\leq \left(\int \log h \, d\nu^- - H(\nu^- \mid m)\right) + \left(\int \log k \, d\nu^+ - H(\nu^+ \mid m)\right).$$

Applying the dual formula to each term yields

$$\log \sum_x f(x) + \log \sum_x g(x) \leq \log \sum_x h(x) + \log \sum_x k(x),$$

which is exactly the Klartag–Lehec inequality after exponentiating.

4.2. The proof strategy

The proof of the entropy theorem is more technical than the hypercube case. The paper first proves an auxiliary multiplicative inequality involving the coupling π and the midpoint measures ν^- and ν^+. Applying the logarithm and then using Jensen's inequality converts this multiplicative estimate into the entropy inequality. The difficult part of the proof is combinatorial. The paper needs to understand how many pairs (x, y) in the support of the monotone coupling can produce the same midpoint. The monotonicity of the coupling gives strong rigidity, and the paper uses this through several lemmas. The main point of the proof is that monotonicity reduces the midpoint structure to something manageable.

5. Curved Brunn–Minkowski Inequalities on $\mathbb{Z}$

The final major contribution of the original paper extends the previous results to curved versions of the discrete Brunn–Minkowski inequalities. While the standard Prékopa–Leindler inequality implicitly assumes a "flat" geometry, the authors demonstrate that introducing a log-concave probability mass function acts as a curved reference measure on $\mathbb{Z}$. By applying the same monotone coupling and discrete transport machinery, they obtain modified displacement convexity bounds. This shows that discrete transport can capture curvature-like phenomena on integer lattices, mirroring the role of Ricci curvature bounds in continuous optimal transport.

6. Conclusion

The paper shows that transport methods can still be effective in discrete settings. On the hypercube, they yield a proof of the Four Functions Theorem through an explicit one-dimensional coupling and induction. On $\mathbb{Z}$, they lead to a displacement convexity of entropy, from which the Klartag–Lehec inequality follows by duality. The most important connection is the role of entropy. Entropy usually appears in tensorization and concentration arguments. In this paper, it appears in a geometric way, and as a quantity that behaves convexly along a transport interpolation.

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