Wide Sets and the Cost of Weakening Replacement

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Abstract

Christopher Menzel's theory of wide sets presents a challenge to the standard relationship between the iterative conception of set and the axiom schema of Replacement. I argue that Menzel's proposal is best understood as a tradeoff between two principles implicit in the iterative conception: availability and closure. While restricting Replacement provides a principled way to accommodate wide sets, doing so weakens a natural closure principle governing the formation of new sets from existing ones. The viability of Menzel's account therefore depends on whether this asymmetry between collection and definable images can be given a positive philosophical justification.

1. Introduction

The iterative conception of set is often taken to provide the intuitive background for standard set theory. On this conception, sets are formed in stages. At the beginning there may be some non-set objects, or atoms; then sets are formed from the objects already available; then further sets are formed from objects available at earlier stages; and so on. This picture is supposed to explain why some collections are sets while others are not. A collection counts as a set only if its members are available together at some stage of the hierarchy.

Christopher Menzel argues that this familiar picture creates a problem for standard Zermelo–Fraenkel set theory with urelements. ZFCU permits atoms, but it does not permit what Menzel calls a "wide" set of atoms: a set containing all atoms, where the atoms are too numerous to be measured by any cardinal number.

Menzel's central thought is that this exclusion is not obviously supported by the iterative conception itself. If all atoms are available at the ground floor of the hierarchy, then why should they not form a set at the first stage? The obstacle to sethood, on this view, is not sheer size, but failure to be bounded in rank. A proper class such as the class of all ordinals is not a set because its members appear at arbitrarily high stages. But the atoms, however many there are, are all present at the base.

This raises a difficult question. If wide sets are allowed, standard ZFCU must be modified. Menzel focuses especially on the axiom schema of Replacement, which says, roughly, that the image of a set under a definable functional operation is again a set. Menzel argues that full Replacement makes wide sets impossible because it builds a limitation-of-size assumption into the concept of set. His proposed solution is to restrict Replacement so that it applies only to narrow, or mathematically determinable, sets.

I argue that Menzel's proposal should be understood as a cost-benefit tradeoff between two ideals within the iterative conception. On the one hand, there is an ideal of availability. Objects already available at a stage should be eligible for collection, supporting Menzel's wide sets. On the other hand, there is an ideal of closure. Once a set exists, definite operations on its members should produce further sets. This ideal supports Replacement. Menzel's restriction of Replacement is therefore not merely an ad hoc technical repair, since it is motivated by a genuine tension in the iterative conception. But it is also not cost-free. It weakens a natural closure principle that seems closely connected to the idea of set formation. The benefit of accommodating wide sets may justify that cost, but only if Menzel can explain why wide sets are collectable without being fully closed under Replacement.

2. Menzel's Motivation for Wide Sets

In ZFCU, atoms are objects that can be members of sets but are not themselves sets and have no members. The usual cumulative hierarchy can therefore be pictured as built over an initial stock of atoms. At the first stage, sets of atoms are formed. At later stages, sets are formed from atoms and from sets formed earlier. This is the impure version of the iterative conception. Menzel asks us to consider the possibility that there are “absolutely many” atoms: more atoms than can be measured by any cardinal number. Let

$$A^* = \{x : x \text{ is an atom}\}.$$

The question is whether A^*, the set of all atoms, can exist. In ordinary ZFCU, the answer is no if there are more atoms than any cardinal number. The reason is that ZFCU proves that every set is equinumerous with some ordinal, and hence that every set has a cardinality. So if A^* were larger than every cardinal, it could not be a set. Menzel thinks this result reveals a disconnect between ZFCU and the iterative conception. The iterative conception does not seem to say that sets must have a cardinality. It seems to say that sets must be formed from objects that are jointly available at some stage. Rank, rather than size, is the central notion. The rank of an atom is 0, since atoms have no members. So if all atoms have rank 0, then the collection of all atoms appears to have rank 1:

$$\text{rank}(a) = 0 \qquad \text{for every atom } a,$$

and therefore

$$\text{rank}(A^*) = 1.$$

This is the intuitive case for wide sets. Even if there are too many atoms to be assigned a cardinality, they are not spread out through the hierarchy. They are all present at the ground floor. By contrast, the class of all ordinals is not a set because its members occur at unboundedly many stages. It is “too high.” The set of all atoms, if it exists, is not too high. It is only “too wide.”

This distinction is philosophically important because it suggests that the usual phrase “too big to be a set” is misleading. If the iterative conception is fundamentally about stages, then the problem with proper classes is not size as such. The problem is that their members are not all available together at any rank. Menzel’s wide set of atoms is different, as its members are all available immediately. So the defender of ordinary ZFCU owes an explanation of why availability is not enough.

3. Replacement and the Source of the Problem

The main technical obstacle to wide sets is Replacement. Informally, Replacement says that if A is a set and each element x of A is assigned exactly one object y, then the collection of all assigned objects is also a set. That is,

$$\forall x \in A\, \exists! y\, \phi(x, y) \quad \Longrightarrow \quad \exists B\, \forall y\, (y \in B \leftrightarrow \exists x \in A\, \phi(x, y))\,.$$

This principle has an obvious intuitive appeal. If A is already a completed set, and if there is a definite rule assigning one object to each member of A, then why should the resulting image fail to be a set? For example, if $A = \omega$, and we define a function that sends each natural number n to some set V_n, then Replacement tells us that

$$\{V_n : n \in \omega\}$$

is a set. This captures a natural closure idea that set-sized operations should produce set-sized outputs.

Menzel’s worry is that this idea becomes dangerous in the presence of wide sets. Suppose A^* is the set of all atoms. If unrestricted Replacement applies to A^*, then any definable functional mapping on A^* must have a set as its range. But if A^* is wider than the entire height of the hierarchy, such a mapping could assign atoms to objects of arbitrarily high rank. Schematically, we might have a rule

$$a \longmapsto y_a,$$

where the objects y_a occur at unboundedly many stages. Unrestricted Replacement would then imply that

$$\{y_a : a \in A^*\}$$

is a set. But this collection would violate the iterative conception, since its members would not be bounded in rank. It would be too high to be formed at any stage.

This is why Menzel claims that Replacement imports a limitation-of-size assumption into set theory. Full Replacement guarantees that the range of any definable mapping on a set is bounded in rank. But in the presence of wide sets, that guarantee connects sethood too closely with narrowness. If every image of every set must be bounded in rank, then wide sets cannot behave as genuinely wide sets. Replacement effectively says that no set can be so wide that it allows one to map out of the hierarchy. Menzel's diagnosis is therefore that ZFCU rules out wide sets not because the iterative conception directly rules them out, but because Replacement strengthens the iterative conception in a particular way. It adds a powerful closure principle, and that principle has consequences for size.

4. Menzel's Restriction of Replacement

Menzel does not simply abandon Replacement. He recognizes that Replacement has substantial theoretical value. To remove it entirely would be too drastic, since set theory is not only meant to describe the minimal content of the iterative conception. It is also meant to provide a fruitful theory of the universe of sets. Some axioms may go beyond what the iterative conception directly entails while still being natural and useful. Instead, Menzel proposes a restricted form of Replacement.

The basic idea is that Replacement should apply only to determinable, or narrow, sets. A set is determinable if it is no larger than some pure set. The pure sets form the inner core of the hierarchy, built from the empty set alone. Since this pure hierarchy is narrow, Replacement cannot create the same problem when applied only to sets comparable in size to pure sets. The restricted principle can be represented informally as follows:

$$\text{If } A \text{ is determinable and } \phi \text{ maps each } x \in A \text{ to a unique } y,$$
$$\text{then } \{y : \exists x \in A\, \phi(x, y)\} \text{ is a set.}$$

This preserves Replacement for ordinary mathematical purposes while blocking its application to wide sets such as A^*. Menzel's thought is that unrestricted Replacement is too strong because it allows wide sets to generate collections that are unbounded in rank. Restricted Replacement avoids this problem. Wide sets can exist, but they are not automatically closed under all definable images.

This is a principled move in at least one sense. Menzel is not merely changing the rules after encountering a contradiction. He identifies a specific feature of full Replacement that conflicts with his interpretation of the iterative conception, and makes narrowness a condition of sethood. If the iterative conception is about rank rather than size, then this feature of Replacement is not obviously justified.

Still, the restriction carries a cost, as Replacement is not just a technical convenience. It expresses the idea that sets are stable under definite operations. If A is genuinely a set, and if a rule assigns exactly one object to each member of A, then it seems natural to think that the image of A under that rule should also be a set. Menzel's view requires us to deny this for wide sets. The set of all atoms may exist, but some definite images of that set may fail to exist as sets.

5. The Cost-Benefit Analysis

The dispute should therefore not be framed as a choice between an ad hoc revision and a perfectly natural orthodoxy. Both sides have costs.

The benefit of ordinary ZFCU is closure, as it preserves the full force of Replacement. Once a set exists, any definable functional image of it is also a set. This makes the universe of sets mathematically stable and theoretically powerful. It also captures a natural thought about completed sets. If their members are available together, then systematically transforming those members should not destroy sethood.

But ordinary ZFCU also has a cost. It excludes wide sets even when the iterative conception seems to allow them. If all atoms are present at the ground floor, it is not clear why they cannot be collected at the first stage. To deny the set of all atoms simply because it lacks a cardinality may look like importing a size limitation that the iterative conception itself does not contain. From Menzel's perspective, ordinary ZFCU privileges narrowness over availability.

Menzel's theory reverses the priority. Its benefit is that it takes availability seriously. Since the atoms are all available at rank 0, their collection is available at rank 1. On this view, the set of all atoms is legitimate because it is bounded in rank, even if it is not bounded in size. This preserves what Menzel takes to be the deepest lesson of the iterative conception that sethood is determined by structure, not by cardinality.

A simple way to see the tradeoff is to imagine a definable assignment from atoms to ranks. Suppose that, for each atom a, there is a rule assigning a to some object y_a, and suppose the assigned objects occur at arbitrarily high stages of the hierarchy. If we accept full Replacement, then the image

$$\{y_a : a \in A^*\}$$

must be a set. This preserves the closure ideal as definite images of sets are sets. But it threatens the availability ideal, because the resulting collection is not bounded in rank and so does not seem available at any stage.

If, on the other hand, we restrict Replacement so that it does not apply to A^*, then this problematic image need not be a set. This preserves the availability ideal that the atoms may be collected because they are all present at the ground floor, while unbounded images are blocked. But it weakens the closure ideal, since there is now a set and a definite operation on its members whose image is not guaranteed to be a set.

Ordinary ZFCU protects closure at the cost of excluding some collections that appear available. Menzel's theory protects availability at the cost of limiting closure. The disagreement is therefore not between a natural theory and an artificial revision, but between two different ways of deciding which part of the iterative conception should have priority when the two come apart.

Menzel must say that not every definite operation on a set produces a set. In particular, wide sets are collectable but not fully closed under Replacement. This is philosophically delicate, because if the atoms are available enough to be collected into A^*, why are they not available enough to support definable images? Menzel can answer that images of A^* may be unbounded in rank, whereas A^* itself is not. But this answer shows that the notion of availability must be more complicated than it first appeared. Availability of members is enough for collection, but not enough for unrestricted transformation.

The choice between these views depends on which ideal better expresses the spirit of the iterative conception. If the core thought is that all objects present at a stage can be collected, then Menzel's theory has a strong claim. But if the core thought is that completed sets should be closed under definite operations, then weakening Replacement is a serious cost. I think the correct conclusion is moderate. Menzel's restriction of Replacement is not merely ad hoc; rather, it is motivated by the philosophical tension we have discussed between the iterative conception and ordinary ZFCU. Full Replacement does seem to add a limitation-of-size principle that is not obviously part of the basic stage picture. However, Menzel's account remains incomplete unless he gives a more positive explanation of why wide sets are collectable but not fully closed under definable images. It is not enough to say that unrestricted Replacement would cause trouble. The restriction must be justified as part of a coherent conception of set formation.

6. Choice and the Unequal Cost of Axiom Revision

This point can be clarified by comparing Replacement with Choice. One might think that if wide sets motivate restrictions on Replacement, they should also motivate restrictions on Choice. But the philosophical situation is different.

Choice says, roughly, that given a set of nonempty sets, there is a function choosing one element from each. The motivation for Choice is not obviously the same as the motivation for Replacement. Choice can be defended as a broadly logical or combinatorial principle: if every set in a family contains something, then there should be a way to select one member from each. Replacement, by contrast, is more directly about set formation. It says that certain collections exist as sets because they arise as images of already existing sets.

This distinction matters because not all axiom revisions are equally costly. Restricting Choice would be one kind of cost. Restricting Replacement is a deeper cost for a theory grounded in the iterative conception, because Replacement concerns the production of new sets from old ones. If Menzel modifies Replacement, he modifies a principle that appears internal to the very idea of the cumulative hierarchy.

This does not refute Menzel. Rather, it helps specify what he must show. He does not need to show that every standard axiom follows from the iterative conception. But he does need to explain why Replacement, despite its naturalness, should be limited to narrow sets. Otherwise, the restriction may seem motivated only by the desire to save wide sets from contradiction.

7. The Boundary of Set Formation

Menzel's theory of wide sets reveals a genuine ambiguity in the iterative conception. On one reading, the iterative conception is primarily a doctrine of availability: sets are formed from objects already present at a stage. On this reading, the set of all atoms is natural, since all atoms are present at the ground floor. On another reading, the iterative conception also supports a doctrine of closure: once a set is formed, definite operations on its members should produce further sets. On this reading, full Replacement is natural.

Menzel's proposal favors the first ideal over the second. He allows wide sets in order to respect the availability of atoms, but he restricts Replacement in order to prevent wide sets from generating collections that are too high. This is not an arbitrary technical repair. It is a serious attempt to separate sethood from cardinality. Still, it comes at a philosophical cost. Menzel must explain why the set of all atoms can be formed even though its definite images need not be.

The best objection to Menzel, then, is not that he modifies Replacement. Any adequate theory of wide sets may need to modify some familiar axioms. The deeper worry is that Menzel has not fully explained the new boundary between legitimate collection and illegitimate closure. His theory is promising because it shows that ordinary ZFCU may not exhaust the iterative conception. But its ultimate success depends on whether the restriction of Replacement can be given a positive philosophical justification, rather than merely a technical role in making room for wide sets.

References

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